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(1 + 2)-Dimensional Model of the Early Universe

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An anisotropic cosmological model is obtained by solving (1 + 3)-dimensional field equations. The topology of the model is $R^1 \otimes M^2 \otimes S^1$, where R^1 is the real line (time axis), M^2 is 2-dimensional space, and S^1 is the circle. Employing the method of Kaluza–Klein type compactification on S^1 and one-loop quantum correction to scalar fields, an effective (1 + 2)-dimensional gravity is obtained. The resulting (1 + 2)-dimensional cosmological model of the early universe is derived.

1. INTRODUCTION

Recently there has been much interest in (1 + 2)-dimensional gravity, as it is supposed to be a useful toy model for a (1 + 3)-dimensional theory of gravitation. Until recently the existence of gravity in (1 + 1)-dimensional space-time and (1 + 2)-dimensional space-time was supposed to be a theory without any intrinsic dynamics. Fujiwara *et al.* (1991) have discussed nucleation of the universe in (1 + 2)-dimensional gravity and topological changes in the realm of quantum gravity. Souradeep and Sahani (1992) have discussed quantum effects near a point in 3-dimensional gravity.

Here, using the method of spontaneous compactification in Kaluza–Klein-type theories (McGuigan, 1991; Srivastava, 1992a,b, 1993) a (1 + 2)-dimensional cosmological model is obtained. This approach is new in the sense that the (1 + 2)-dimensional cosmological model is obtained from the (1 + 3)-dimensional anisotropic model of the early universe without a "crack of doom" singularity. Earlier, this method was employed (McGuigan, 1991; Srivastava, 1993) to get (1 + 1)-dimensional gravity. In the present paper, an anisotropic singularity-free (1 + 3)-dimensional cosmological model is obtained by solving the Einstein field equations. The topology of this model

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is given as $R^1 \otimes M^2 \otimes S^1$ (R^1 is the real line, which is the time axis, M^2 is 2-dimensional space, and S^1 is the circle, which is compact). The isometry group $U(1)$ acts transitively on the compact manifold S^1. Kaluza–Klein-type compactification is done on S^1. As a result, a (1 + 2)-dimensional cosmological model is obtained.

The focus of this paper is first getting a (1 + 3)-dimensional cosmological model by solving Einstein's field equations exactly. Then it is discussed that a physically meaningful model will be spatially flat. Second, using Kaluza–Klein compactification, (1 + 2)-dimensional gravity is obtained. The paper is organized as follows. Section 2 contains the exact solution of Einstein's field equations. In Section 3, dimensional reduction of scalar as well as gravitational fields is discussed. Section 4 contains the one-loop correction to dimensionally reduced scalar fields. In Section 5, an effective action for (1 + 2)-dimensional gravity is obtained. Section 6 is a concluding section which discusses some cosmological implications of the (1 + 2)-dimensional model. Natural units ($\hbar = c = 1$) are used throughout the paper.

2. (1 + 3)-DIMENSIONAL ANISOTROPIC COSMOLOGICAL MODEL

The cosmological model having topology $R^1 \otimes M^2 \otimes S^1$ has the line element

$$ds^2 = dt^2 - a^2(t)\left(\frac{dr^2}{1 - k_2 r^2} + r^2\, d\theta^2\right) - b^2(t)\rho^2\, d\theta_E^2 \tag{2.1}$$

where t is the cosmic time, $a(t)$ and $b(t)$ are scale factors, ρ is the radius of S^1 (circle), k_2 is the spatial curvature with possible values +1, 0, −1 for closed, flat, and open spatial submanifold M^2, respectively, and $0 \le \theta_E \le 2\pi$.

The energy-momentum tensor for the anisotropic fluid can be written as

$$T_{\mu\nu} = (\epsilon + p)u_\mu u_\nu - (\delta p + \tilde{\delta}\tilde{p})g_{\mu\nu} \tag{2.2}$$

where $\mu, \nu = 0, 1, 2, 3$; ϵ is the energy density, p is the pressure on M^2, $\tilde{p}$ is the pressure on the compact manifold S^1, and $\tilde{\delta} = 1 - \delta$ with

$$\delta = \begin{cases} 1 & \text{for} \quad \mu, \nu = 0, 1, 2 \\ 0 & \text{for} \quad \mu, \nu = 3 \end{cases}$$

Thus,

$$T^0_0 = \epsilon, \qquad T^1_1 = T^2_2 = -p, \qquad T^3_3 = -\tilde{p} \tag{2.3}$$

In the background geometry with the line element given by equation (2.1), Einstein's field equations are

$$\frac{1}{2} G_0^0 = \frac{k_2}{a^2} + \left(\frac{a'}{a}\right)^2 + 2\frac{a'}{a}\frac{b'}{b} = 4\pi G t_P^2 \epsilon \tag{2.4a}$$

$$\frac{2k_2}{a^2} + \frac{d}{d\tau}\left(\frac{a'}{a}\right) + \frac{a'}{a}\left(2\frac{a'}{a} + \frac{b'}{b}\right) = -4\pi G t_P^2(\epsilon - \tilde{p}) \tag{2.4b}$$

$$\frac{d}{d\tau}\left(\frac{b'}{b}\right) + \frac{b'}{b}\left(2\frac{a'}{a} + \frac{b'}{b}\right) = -4\pi G t_P^2(\epsilon + \tilde{p} - 2p) \tag{2.4c}$$

where prime denotes differentiation with respect to the dimensionless parameter $\tau = t/t_P$ (t_P is the Planck time), G stands for the four-dimensional Newtonian gravitational constant, and G_ν^μ are components of the Einstein tensor. We have three constraint equations:

$$G^\mu_{\nu;\mu} = 0 \tag{2.5a}$$

$$T^\mu_{\nu;\mu} = 0 \tag{2.5b}$$

$$G^\mu_{\nu;\mu} = T^\mu_{\nu;\mu} = 0 \tag{2.5c}$$

where the semicolon stands for covariant differentiation.

In the geometry given by equation (2.1), equation (2.5a) yields the constraint equation

$$(G_0^0)' = 0 \tag{2.6a}$$

equations (2.2) and (2.5b) imply that

$$\epsilon' + \epsilon\left(2\frac{a'}{a} + \frac{b'}{b}\right) + 2p\frac{a'}{a} + \tilde{p}\frac{b'}{b} = 0 \tag{2.6b}$$

and equation (2.5c) yields

$$G_0^0 = T_0^0 \text{ const} \tag{2.6c}$$

which reduces to equation (2.4a) on using the definitions of G_0^0 and T_0^0. Solutions of equations (2.4b) and (2.4c) should satisfy the constraint equations (2.6a)–(2.6c) at all times. If these equations are satisfied at one particular time, these can be treated as satisfied at all times. So, for convenience, one can choose the particular epoch $\tau = 0$ and can find conditions obeying the constraint equations (2.6).

Using the conditions

$$p = \epsilon + \frac{k_2}{4\pi G t_P^2 a^2} \tag{2.7a}$$

and

$$\tilde{p} = \epsilon + \frac{k_2}{2\pi G t_P^2 a^2} \tag{2.7b}$$

in (2.4b) and (2.4c), one obtains

$$\frac{d}{d\tau}\left(\frac{a'}{a}\right) + \frac{a'}{a}\left(2\frac{a'}{a} + \frac{b'}{b}\right) = 0 \tag{2.8a}$$

$$\frac{d}{d\tau}\left(\frac{b'}{b}\right) + \frac{b'}{b}\left(2\frac{a'}{a} + \frac{b'}{b}\right) = 0 \tag{2.8b}$$

Now, it is helpful to make the ansatz

$$b^2 = f^2 + \frac{1}{a^2(\tau)} \tag{2.9}$$

Using the ansatz given by (2.9) in (2.8a), one obtains

$$a'(1 + f^2)^{1/2} a^2 = \tilde{A} \tag{2.10}$$

where $\tilde{A}$ is an integration constant. Connecting equations (2.8b) and (2.9), one obtains

$$\frac{a'}{(1 + f^2 a^2)^{1/2}} = \tilde{B} \tag{2.11}$$

Equations (2.10) and (2.11) yield

$$a' = (\tilde{A}\tilde{B})^{1/2} \tag{2.12}$$

Now rescaling a to $a(\tilde{A}\tilde{B})^{1/2}$, one obtains

$$a' = 1 \tag{2.13}$$

which yields the solution

$$a = a_0 + \tau \tag{2.14}$$

where $a_0 = a(\tau = 0)$.

Using equations (2.7) and (2.14) in equation (2.6b), we obtain

$$\epsilon = (a_0 + \tau)^{-2}[1 + f^2(a_0 + \tau)^2]^{-2} \times \left[A - \frac{k_2 f^2}{2\pi G t_P^2} \ln\left(\frac{a_0 + \tau}{[1 + f^2(a_0 + \tau)^2]^{1/2}}\right)\right] \tag{2.15}$$

Using the results given by (2.14) and (2.15) in the constraint equation (2.4a), we can evaluate the integration constant A as

$$A = (4\pi G t_P^2)^{2-1}\{(k_2 t_P^2 + 1)(1 + f^2 a_0^2)^2 - 2(1 + f^2 a_0^2) + 2k_2 f^2 \ln[a_0/(1 + f^2 a_0^2)^{1/2}]\} \tag{2.16}$$

Thus, from equations (2.15) and (2.16)

$$\epsilon = (4\pi G t_P^2)^{-1}(a_0 + \tau)^{-2}[1 + f^2(a_0 + \tau)^2]^{-2} \times \left\{(k_2 t_P^2 + 1)(1 + f^2 a_0^2)^2 - 2(1 + f^2 a_0^2) + 2k_2 f^2 \ln\left(\frac{a_0}{a_0 + \tau}\right)\left[\frac{1 + f^2(a_0 + \tau)^2}{1 + f^2 a_0^2}\right]^{1/2}\right\} \tag{2.17}$$

From equation (2.17), one can derive the following conclusions:

1. If $k_2 = 0$, $a_0 \neq 0$,

$$\epsilon = \frac{(1 + f^2 a_0^2)(f^2 a_0^2 - 1)}{4\pi G t_P^2 (a_0 + \tau)^2[1 + f^2(a_0 + \tau)^2]^2} \tag{2.18}$$

Since ϵ is the energy density, it will be positive. So, in this case

$$f^2 a_0^2 > 1$$

2. If $k_2 = 0$, $a_0 = 0$, $\epsilon < 0$ at all epochs, which is unphysical.
3. If $k_2 = \pm 1$, $a_0 = 0$, ϵ will be divergent at all times.
4. If $k_2 = +1$, $a_0 \neq 0$,

$$\epsilon = (4\pi G t_P^2)^{-1}(a_0 + \tau)^{-2}[1 + f^2(a_0 + \tau)^2]^{-2} \times \left\{(1 + f^2 a_0^2)[(1 + t_P^2)(1 + f^2 a_0^2) - 2] + 2f^2 \ln\left(\frac{a_0}{a_0 + \tau}\right)\left[\frac{1 + f^2(a_0 + \tau)^2}{1 + f^2 a_0^2}\right]^{1/2}\right\} \tag{2.19}$$

5. If $k_2 = -1$, $a_0 \neq 0$,

$$\epsilon = (4\pi G t_P^2)^{-1}(a_0 + \tau)^{-2}[1 + f^2(a_0 + \tau)^2]^{-2} \times \left\{(1 + f^2 a_0^2)[(1 - t_P^2)(1 + f^2 a_0^2) - 2] + 2f^2 \ln\left(\frac{a_0 + \tau}{a_0}\right)\left[\frac{1 + f^2 a_0^2}{1 + f^2(a_0 + \tau)^2}\right]^{1/2}\right\} \tag{2.20}$$

Looking at the above five cases, one finds that the constraint equation (2.6c) is satisfied at $\tau = 0$ if $a_0 \neq 0$ and $k_2 = 0$ or ± 1. The constraint equation (2.6a) yields the following results:

1. $f^2 a_0^2 = 1 + \sqrt{2}$ if $k_2 = 0$.
2. $f^2 2a_0^2 = -1/2$ if $k_2 = -1$.
3. $f^2 a_0^2 = 0$ if $k_2 = +1$.

Since f is a real number, $f^2 a_0^2 = -1/2$ implies that a_0 should be complex, which is unphysical. This indicates that $k_2 \neq -1$. Now, f cannot be zero, because the vanishing of f implies the existence of a "crack of doom" singularity. So, if $k_2 = +1$, $a_0 = 0$. But if $a_0 = 0$, the constraint equation (2.6c) will not be satisfied at any time. This indicates that the choice $k_2 = +1$ also is not possible. Ultimately, one finds that the only possibility is $k_2 = 0$ with

$$f^2 a_0^2 = 1 + \sqrt{2} \tag{2.21}$$

Thus, one finds the solution of equations (2.4) as

$$a = a_0 + (t/t_P) \tag{2.22}$$

and using equation (2.14) in equation (2.9), we have

$$b^2 = f^2 + t_P^2(a_0 t_P + t)^{-2} \tag{2.23}$$

According to the discussions given above, one finds that $k_2 = 0$ and $a_0 \neq 0$. In the case $a_0 = 0$, some constraint equations are not obeyed. So, to get the cosmological model obeying equations (2.4) at all epochs ($t \geq 0$), a_0 should be nonzero and positive, which is given by equation (2.21) provided that f is evaluated. Evaluation of f will be discussed later. A nonzero positive value of a_0 implies a singularity-free cosmological model.

Because of the Hawking–Penrose theorem, the big-bang singularity might be supposed to be inescapable in general relativity. But this supposition was shaken in 1990 due to the discovery of the singularity-free cosmological solution of general relativity by Senovilla (1990). Later, other authors also obtained some interesting cosmological solutions without a singularity (Ruiz and Senovilla, 1992; Dadhich and Patel, 1993). It is appropriate to mention here that the Hawking–Penrose theorem was proved for closed models or models with closed, trapped surfaces (Hawking and Ellis, 1973). The cosmological models mentioned above (Senovilla, 1990; Ruiz and Senovilla, 1992; Dadhich and Patel, 1993) and the model derived here are neither closed nor contain any closed, trapped surface. So, acceptance of this theorem does not make these singularity-free cosmological models invalid.

Thus, the (3 + 1)-dimensional anisotropic singularity-free cosmological model of the early universe is obtained as

$$ds^2 = dt^2 - a^2(t)(dr^2 + r^2\, d\theta^2) - b^2(t)\rho^2\, d\theta_E^2 \tag{2.24}$$

with $a(t)$ and $b(t)$ given by equations (2.22) and (2.23), respectively. From here on, we work with this line element.

Using equation (2.22) in equation (2.18), we obtain

$$\epsilon = \frac{2(1+\sqrt{2})^3\epsilon_0}{[f\tau + (1+\sqrt{2})^{1/2}]^2\{1 + [f\tau + (1+\sqrt{2})^{1/2}]^2\}^2} \tag{2.25a}$$

with

$$\epsilon_0 = \frac{\sqrt{2}\, M_P^4 f^2}{4\pi(1+\sqrt{2})^2\sqrt{2}} \tag{2.25b}$$

$b(t)$, given by equation (2.23), does not have a "crack-of-doom" singularity,

$$\lim_{t\to\infty} b(t) = f > 0$$

3. DIMENSIONAL REDUCTION

3.1. Gravity

The four-dimensional action for gravity is given by

$$S_g^{(4)} = -\frac{1}{16\pi G}\int d^4x(-g_4)^{1/2}R_4 \tag{3.1}$$

where G is the $(1 + 3)$-dimensional gravitational constant. g_4 is the determinant of $g_{\mu\nu}$ and R_4 is the Ricci scalar obtained from $g_{\mu\nu}$.

For the sake of convenience, the metric tensor given by equation (2.24) is written as

$$g_{\mu\nu} = \begin{pmatrix} g_{\mu'\nu'} & 0 \\ 0 & -b^2(t)\rho^2 \end{pmatrix} \tag{3.2}$$

where $g_{\mu'\nu'} = \text{diag}(1, -a^2, -a^2)$. Now $g_{\mu\nu}$ is conformally transformed to $\tilde{g}_{\mu\nu}$ as

$$g_{\mu\nu} = b^2(t)\tilde{g}_{\mu\nu} = b^2(t)\begin{pmatrix} \tilde{g}_{\mu'\nu'} & 0 \\ 0 & -\rho^2 \end{pmatrix} \tag{3.3}$$

where

$$\tilde{g}_{\mu'\nu'} \equiv \text{diag}(b^{-2}, -a^2b^{-2}, -a^2b^{-2})$$

Now equation (3.1) can be rewritten as

$$S_g^{(4)} = -\frac{1}{16\pi G}\int d^3x\, d\theta_E\, b^2\rho(\tilde{g}_3)^{1/2}\left[\tilde{R}_3 - 18\left(\frac{db}{dt}\right)^2\right] \tag{3.4}$$

Ignoring terms of total divergence and integrating over θ_E, one obtains

$$S_g^{(3)} + S_{\text{ind}}^{(3)(m)} = -\frac{\rho}{8G}\int d^3x\, b^2(\tilde{g}_3)^{1/2}\left[b^2R_3 - 22\left(\frac{db}{dt}\right)^2\right] \tag{3.5}$$

To undo the earlier conformal transformation, another conformal transformation

$$\tilde{g}_{\mu'\nu'} = b^{-2}g_{\mu'\nu'} \tag{3.6}$$

is employed. As a result, from equation (3.5), one obtains

$$S_g^{(3)} + S_{\text{ind}}^{(3)(m)} = -\frac{1}{16\pi G_3}\int d^3x\, a^2b\left[R_3 - \frac{22}{b^2}\left(\frac{db}{dt}\right)^2\right] \tag{3.7}$$

where $G_3 = G/2\pi\rho$ and

$$S_{\text{ind}}^{(3)(m)} = \int d^3x\, a^2\left[\frac{11}{8\pi G_3 b}\left(\frac{db}{dt}\right)^2\right] \tag{3.8}$$

which is a contribution to the matter fields induced by the compactification of the circular component of space.

3.2. Scalar Fields

The existence of some scalar field ϕ with bare mass m_0 is assumed in the background geometry with the action given by

$$S_\phi^{(4)} = \frac{1}{2}\int d^3x\, d\theta_E\, a^2b\rho[g^{\mu\nu}\partial_\mu\phi^*\partial_\nu\phi - (\xi R_4 + m_0^2)\phi^*\phi] \tag{3.9}$$

where ξ is a nonminimal coupling constant and

$$R_4 = R_3 - \frac{22}{b^2}\left(\frac{db}{dt}\right)^2 \tag{3.10a}$$

with

$$R_3 = 4a^{-1}\frac{d^2a}{dt^2} \tag{3.10b}$$

On the space-time with topology $R^1 \otimes M^2 \otimes S^1$, ϕ can be decomposed as

$$\phi = (2\pi\rho b)^{-1/2}\sum_{n=-\infty}^{\infty}\phi_n(t, r, \theta)\exp[i(n+\alpha)\theta_E] \tag{3.11}$$

where $\alpha = 0$ (1/2) for untwisted (twisted) fields. S^1 is a manifold which is not simply connected, so the possibility exists for untwisted (twisted) fields on the circle. From here on, only untwisted scalar fields on S^1 will be considered (Srivastava, 1992b, 1993).

Substituting the decomposed form of ϕ given by equation (3.11) into equation (3.9) and integrating over θ_E, one obtains

$$S_\phi^{(3)} = -\frac{1}{2}\int d^3x\, a^2 \sum_{n=-\infty}^{\infty} \phi_n^* \left[\frac{1}{a^2}\frac{\partial}{\partial t}\left(a^2 \frac{\partial}{\partial t}\phi_n\right) - \frac{1}{a^2 r}\frac{\partial}{\partial r}\left(r\frac{\partial \phi_n}{\partial r}\right) - \frac{1}{a^2 r^2}\frac{\partial^2 \phi_n}{\partial \theta^2} + m_n^2 \phi_n\right] \tag{3.12a}$$

where

$$m_n^2 = \tilde{m}_0^2 + \frac{n^2}{\rho^2 b^2} \tag{3.12b}$$

with

$$\tilde{m}_0^2 = m_0^2 + \xi R_3 - \frac{22\xi}{b^2}\left(\frac{db}{dt}\right)^2 - \frac{1}{2b}\frac{d^2 b}{dt^2} + \frac{3}{4b^2}\left(\frac{db}{dt}\right)^2 \tag{3.12c}$$

4. ONE-LOOP QUANTUM CORRECTION TO ϕ_n

Here the one-loop quantum correction to the scalar fields ϕ_n is obtained by employing the operator regularization method, which is an extension of zeta-function regularization.

Now, ϕ_n is a 3-dimensional scalar field. So, on adding the significant contribution of the one-loop correction, the effective action is obtained up to adiabatic order 4 as (Mann *et al.*, 1988/89)

$$\Gamma = S_{0\phi}^{(3)} + \sum_{n=-\infty}^{\infty} \frac{d}{ds}\left(\frac{(s+1/2)^{1/2}}{\Gamma_s (4\pi)^{3/2}}\left(\frac{\mu^2}{m_n^2}\right)^s \times \int d^3x\, a^2 \left\{\frac{(m_n^2)^{3/2}}{(s-3/2)(s-1/2)} + \frac{(m_n^2)^{1/2}}{(s-1/2)}\left(\frac{1}{6}-\xi\right)R_3 + (m_n^2)^{-1/2}\left[\frac{1}{30}\Box_3 R_3 + \frac{1}{180} R_3^{\mu'\nu'\alpha'\beta'} R_{3\mu'\nu'\alpha'\beta'} - \frac{1}{180} R_3^{\mu'\nu'} R_{3\mu'\nu'} - \frac{1}{6}\xi\,\Box_3 R_3 + \frac{1}{2}\left(\xi - \frac{1}{6}\right)^2 R_3^2\right]\right\}\right)\Bigg|_{s=0}$$

$$(\mu', \nu', \alpha', \beta', \ldots = 0, 1, 2) \tag{4.1}$$

Now, Γ_s has a pole at $s = 0$, so one can write

$$\Gamma_s = \frac{1}{s} - \gamma + O(s) \tag{4.2}$$

where $\gamma = 0.5772$ is the Euler constant. Now, using equation (4.2) in (4.1), we obtain

$$\begin{aligned}\Gamma = S^{(3)}_{0\phi} + (4\pi)^{3/2} \sum_{n=-\infty}^{\infty} \frac{d}{ds}\left(\frac{s(s+1/2)^{1/2}}{1-\gamma s + sO(s)}\left(\frac{\mu^2}{m_n^2}\right)^2\right. \\ \times \int d^3x\, a^2\left\{\frac{m_n^3}{(s-3/2)(s-1/2)} + \frac{m_n}{(s-1/2)}\left(\frac{1}{6}-\xi\right)R_3\right. \\ + m_n^{-1}\left[\frac{1}{6}\left(\frac{1}{5}-\xi\right)\Box_3 R_3 + \frac{1}{180} R_3^{\mu'\nu'\alpha'\beta'} R_{3\mu'\nu'\alpha'\beta'} - \frac{1}{180} R_3^{\mu'\nu'} R_{3\mu'\nu'}\right. \\ \left.\left.\left.+ \frac{1}{2}\left(\xi - \frac{1}{6}\right)^2 R_3^2\right]\right\}\right)\Bigg|_{s=0}\end{aligned} \tag{4.3}$$

Using the Riemann zeta function

$$\rho(r) = \sum_{n=1}^{\infty} \frac{1}{n^r} \tag{4.4}$$

one can easily obtain from equation (4.3)

$$\begin{aligned}\sum_{n=-\infty}^{\infty} (m_n^2)^{3/2-s} \\ = (\tilde{m}_0^2)^{3/2-s} \sum_{n=-\infty}^{\infty} 1 + \left(\frac{3}{2}-s\right)(\tilde{m}_0^2)^{1/2-s}\rho^{-2}b^{-2} \sum_{n=-\infty}^{\infty} n^2 \\ + \frac{1}{2}\left(\frac{3}{2}-s\right)\left(\frac{1}{2}-s\right)(\tilde{m}_0^2)^{-1/2-s}\rho^{-2}b^{-2} \sum_{n=-\infty}^{\infty} n^4 + \cdots \\ = 2(\tilde{m}_0^2)^{3/2-s}\rho(0) + 2(3/2-s)(\tilde{m}_0^2)^{1/2-s}\rho^{-2}b^{-2}\rho(-2) \\ + (3/2-s)(1/2-s)(\tilde{m}_0^2)^{-1/2-s}\rho^{-2}b^{-2}\rho(-4) + \cdots\end{aligned} \tag{4.5}$$

Though $\rho(-2x)$ (x is a positive integer) and $\rho(0)$ are divergent, using the method of analytic continuation one obtains (Bateman, 1955)

$$\rho(0) = -\frac{1}{2} \qquad \text{and} \qquad \rho(-2x) = 0 \qquad \text{for} \quad x > 0$$

As a result, equation (14.5) yields

$$\sum_{n=-\infty}^{\infty} (m_n^2)^{3/2-s} = -(\tilde{m}_0^2)^{3/2-s} \tag{4.6a}$$

Similarly,

$$\sum_{n=-\infty}^{\infty} (m_n^2)^{1/2-s} = -(\tilde{m}_0^2)^{1/2-s} \tag{4.6b}$$

$$\sum_{n=-\infty}^{\infty} (m_n^2)^{-1/2-s} = -(\tilde{m}_0^2)^{-1/2-s} \tag{4.6c}$$

From equations (4.3) and (4.6)

$$\begin{aligned}
\Gamma = S_{0\phi}^{(3)} &+ (4\pi)^{-3/2} \frac{d}{ds}\left(\frac{s(s+1/2)^{1/2}}{1-\gamma s + sO(s)}\left(\frac{\mu^2}{\tilde{m}_0^2}\right)^s\right. \\
&\times \int d^3x\, a^2 \left\{ -\frac{\tilde{m}_0^3}{(s-3/2)(s-1/2)} - \frac{\tilde{m}_0}{(s-1/2)}\left(\frac{1}{6}-\xi\right)R_3\right. \\
&- \tilde{m}_0^{-1}\left[\frac{1}{6}\left(\frac{1}{5}-\xi\right)\Box_3 R_3 + \frac{1}{180} R_3^{\mu'\nu'\alpha'\beta'} R_{3\mu\nu\alpha\beta}\right. \\
&\left.\left.\left.- \frac{1}{180} R_3^{\mu'\nu'} R_{3\mu'\nu'} + \frac{1}{2}\left(\xi - \frac{1}{6}\right)^2 R_3^2\right]\right\}\right)\Bigg|_{s=0} \\
= S_{0\phi}^{(3)} &- \frac{1}{8\pi}\int d^3x\, a^2 \left\{\frac{4\tilde{m}_0^3}{3} - 2\tilde{m}_0\left(\frac{1}{6}-\xi\right)R_3\right. \\
&+ \tilde{m}_0^{-1}\left[\frac{1}{6}\left(\frac{1}{5}-\xi\right)\Box_3 R_3 + \frac{1}{180} R_3^{\mu'\mu'\alpha'\beta'} R_{3\mu'\nu'\alpha'\beta'}\right. \\
&\left.\left.- \frac{1}{180} R_3^{\mu'\nu'} R_{3\mu'\nu'} + \frac{1}{2}\left(\xi - \frac{1}{6}\right)^2 R_3^2\right]\right\}
\end{aligned} \tag{4.7}$$

Equation (4.7) gives the renormalized effective action up to the one-loop correction in (1 + 2)-dimensional space. It is interesting to see that the one-loop correction to ϕ_n contributes to gravity the induced terms

$$\frac{1}{16\pi G_{(3)\text{ind}}} = \frac{1}{4\pi}\left(\xi - \frac{1}{6}\right)\tilde{m}_0 \tag{4.8a}$$

$$\frac{\Lambda_{\text{ind}}}{8\pi G_{(3)\text{ind}}} = -\frac{\tilde{m}_0^3}{6\pi} \tag{4.8b}$$

and the higher derivative terms

$$\chi_{\text{ind}} = \tilde{m}_0^{-1}\left[\frac{1}{6}\left(\frac{1}{5} - \xi\right)\Box_3 R_3 + \frac{1}{180} R_3^{\mu'\nu'\alpha'\beta'} R_{3\mu'\nu'\alpha'\beta'} - \frac{1}{180} R_3^{\mu'\nu'} R_{3\mu'\nu'} + \frac{1}{2}\left(\xi - \frac{1}{6}\right)^2 R_3^2\right] \tag{4.8c}$$

5. EFFECTIVE FUNDAMENTAL CONSTANTS

Including the contribution of the one-loop correction to (1 + 2)-dimensional gravity from equation (4.7) yields the effective gravitational action

$$S^{(3)}_{g,\text{eff}} = -\int d^3x\, a^2\left[-\frac{11}{8\pi G_3 b^2}\left(\frac{db}{dt}\right)^2 + \frac{\tilde{m}_0^3}{6\pi} + \left\{\frac{b}{16\pi G_3} + \frac{\tilde{m}_0}{4\pi}\left(\xi - \frac{1}{6}\right)\right\} R_3 + \frac{\chi_{\text{ind}}}{8\pi\tilde{m}_0}\right] \tag{5.1}$$

where $\tilde{m}_0$ is given by equation (3.12c) and χ is defined by equation (4.8c). Thus from equation (5.1) one obtains

$$\frac{1}{16\pi G_{(3)\text{eff}}} = \frac{b}{16\pi G_3} + \frac{\tilde{m}_0}{4\pi}\left(\xi - \frac{1}{6}\right) \tag{5.2a}$$

where $G_{(3)\text{eff}}$ is the 3-dimensional effective gravitational constant, which is time dependent. The effective time-dependent cosmological constant is given as

$$\frac{\Lambda_{\text{eff}}}{8\pi G_{(3)\text{eff}}} = \frac{11}{8\pi G_3 b^2}\left(\frac{db}{dt}\right)^2 - \frac{\tilde{m}_0^3}{6\pi} \tag{5.2b}$$

Using $b(t)$ from equation (2.23) in equation (5.2a), we obtain

$$\frac{1}{16\pi G_{(3)\text{eff}}} = \frac{[f^2 + t_P^2(a_0t_P + t)^{-2}]^{1/2}}{16\pi G_3} + \frac{1}{4\pi}\left(\xi - \frac{1}{6}\right)\left[m_0^2 + \xi R_3 - \frac{22\xi}{b^2}\left(\frac{db}{dt}\right)^2 - \frac{1}{2b}\frac{d^2b}{dt^2} + \frac{3}{4b^2}\left(\frac{db}{dt}\right)^2\right]^{1/2} \tag{5.3}$$

where R_3 is given by equation (3.10b), and $a(t)$ and $b(t)$ are given by equations (2.22) and (2.23). Taking the limit $t \to \infty$ in equation (5.3), one finds that

$$\frac{1}{16\pi G_{(3)\text{eff}}} = \frac{f}{16\pi G_3} + \frac{m_0}{4\pi}\left(\xi - \frac{1}{6}\right) \tag{5.4}$$

To determine f, one can go back to 4-dimensional gravity, where we know that at late times the gravitational constant is the Newtonian gravitational constant. One can then use $G_3 = G/2\pi\rho$ (where G is the 4-dimensional Newtonian gravitational constant) given in equation (8.7). Thus, one obtains

$$\frac{1}{16\pi G_{\text{eff}}} = \frac{f}{16\pi G} + \frac{m_0}{8\pi^2\rho}\left(\xi - \frac{1}{6}\right) \tag{5.5}$$

As late times, G_{eff} is supposed to be equal to G, so one gets from equation (5.5)

$$f = 1 + \frac{2Gm_0}{\pi\rho}\left(\frac{1}{6} - \xi\right) \tag{5.6}$$

Equation (5.6) implies that

(i) $f = 1$ if $m_0 = 0$ or $\xi = 1/6$ or $m_0 = 0$ and $\xi = 1.6$.
(ii) $f = 0$ if $\xi = 1/6 + \pi\rho/(2Gm_0)$ and $m_0 \neq 0$.

The second implication is not physically valid, as it would mean that at late times the cosmological model with line element given by (2.24) will be (1 + 2)-dimensional. This is not correct, as the present universe is (1 + 3)-dimensional.

The effective cosmological constant is give by equation (5.2b) as

$$\Lambda_{\text{eff}} = 8\pi G_{(3)\text{eff}}\left[\frac{11}{8\pi G_3 b^2}\left(\frac{db}{dt}\right)^2 - \frac{\tilde{m}_0^3}{6\pi}\right]$$
$$= \frac{16\pi G}{4\pi b\rho + 8\tilde{m}_0 G(\xi - 1/6)}\left[\frac{11\rho}{4Gb^2}\left(\frac{db}{dt}\right)^2 - \frac{\tilde{m}_0^3}{6\pi}\right] \tag{5.7}$$

At late times, one obtains from equation (5.7) that

$$\lim_{t\to\infty} \Lambda_{\text{eff}} = -\frac{8m_0^3 G}{3[4\pi f\rho + 8m_0 G(\xi - 1/6)]}$$
$$= -\frac{8m_0^3}{3[4\pi f\rho M_P^2 + 8m_0(\xi - 1/6)]} \tag{5.8}$$

m_0 is the mass of the scalar field ϕ in $(1 + 3)$-dimensional space (used in Section 3). For a physically relevant theory m_0 cannot be greater than the Planck mass M_P. Normally, m_0 is supposed to be quite less than M_P. So from equation (5.8) one finds $\lim_{t\to\infty} \Lambda_{\text{eff}}$ very small.

6. (1 + 2)-DIMENSIONAL GRAVITATIONAL EQUATION AND COSMOLOGY

The $(1 + 2)$-dimensional effective action for gravity is obtained from equations (3.7) and (4.7) as

$$\begin{aligned} &S_g^{(3)} + S_{\text{ind}}^{(3)m} + \Gamma + S_m^{(3)} \\ &= S_{0\phi}^{(3)} + S_m^{(3)} - \frac{1}{8\pi}\int d^3x\,\sqrt{g_3} \\ &\times \left\{\frac{b}{2G_3}R_3 - \frac{11}{G_3 b}\left(\frac{db}{dt}\right)^2 + \frac{4}{3}\tilde{m}_0^3 - 2\tilde{m}_0\left(\frac{1}{6} - \xi\right)R_3\right. \\ &+ \tilde{m}_0^{-1}\left[\frac{1}{6}\left(\frac{1}{5} - \xi\right)\Box_3 R_3 + \frac{1}{180}\right. \\ &\times (R_3^{\mu'\nu'\alpha'\beta'}R_{3\mu'\nu'\alpha'\beta'} - R_3^{\mu'\nu'}R_{3\mu'\nu'}) \\ &\left.\left. + \frac{1}{2}\left(\xi - \frac{1}{6}\right)^2 R_3^2\right]\right\} \end{aligned} \tag{6.1}$$

where $S_m^{(3)}$ is the action of the matter present other than the scalar fields ϕ_n and $\sqrt{g_3} = a^2$ in the effective $(1 + 2)$-dimensional cosmological model

$$ds^2 = dt^2 - a^2(t)(dr^2 + r^2\,d\theta^2) \tag{6.2}$$

with $a(t) = a_0 + (t/t_P)$ given by equation (2.22).

The $(1 + 2)$-dimensional gravitational field equations are derived from equation (6.1) as

$$\left[\frac{b}{2G_3} - 2\tilde{m}_0\left(\frac{1}{6} - \xi\right)\right]\left(R_{(3)\mu'\nu'} - \frac{1}{2} g_{\mu'\nu'} R_3\right) + \frac{1}{2}\left[\frac{11}{G_3 b}\left(\frac{db}{dt}\right)^2 - \frac{4\tilde{m}_0^3}{3}\right] g_{\mu'\nu'}$$
$$+ \frac{1}{2\tilde{m}_0}\left(\xi - \frac{1}{6}\right)^2\left(2R_{3;\mu'\nu'} - 2g_{\mu'\nu'}\Box_3 R_3 - \frac{1}{2} g_{\mu'\nu'} R_3^2 + 2R_3 R_{3\mu'\nu'}\right)$$
$$+ \frac{1}{180\tilde{m}_0}\left(- \frac{1}{2} g_{\mu'\nu'} R_3^{\alpha'\beta'\gamma'\delta'} R_{3\alpha'\beta'\gamma'\delta'} + 2R_{3\mu'\alpha'\beta'\gamma'} R_{3\nu'}^{\alpha'\beta'\gamma'} - 3\,\Box_3 R_{3\mu'\nu'}\right.$$
$$+ 2R_{3;\mu'\nu'} - 4R_{3\mu'\alpha'} R_{3\nu'}^{\alpha'} + 4R_3^{\alpha'\beta'} R_{3\alpha'\mu'\beta'\nu'} - 2R_{3\mu';\nu'\alpha'}^{\alpha'}$$
$$\left. + \frac{1}{2} g_{\mu'\nu'}\Box_3 R_3 - 2R_{3\mu'}^{\alpha'} R_{3\alpha'\nu'} + \frac{1}{2} g_{\mu'\nu'} R_3^{\alpha'\beta'} R_{3\alpha'\beta'}\right) = -8\pi\langle T_{\mu'\nu'}\rangle \quad (6.3)$$

If the matter fields in the model behave like a perfect fluid (which is very likely in the very early universe), we know that

$$\bar{\rho} = \langle T_0^0\rangle, \qquad p_1 = \langle T_1^1\rangle, \qquad p_2 = \langle T_2^2\rangle$$

where $\bar{\rho}$ is the energy density, $\langle\ldots\rangle$ denotes the vacuum expectation value, and p_1 and p_2 are components of the pressure. Thus, in the model given by equation (6.2),

$$\bar{\rho} = \left[\frac{b}{2G_3} - 2\tilde{m}_0\left(\frac{1}{6} - \xi\right)\right]\left(R_{30}^0 - \frac{1}{2} R_3\right) + \frac{1}{2}\left[\frac{11}{G_3 b^2}\left(\frac{db}{dt}\right)^2 - \frac{4\tilde{m}_0^3}{3}\right]$$
$$+ \frac{1}{2\tilde{m}_0}\left(\xi - \frac{1}{6}\right)^2\left(2R_{3;0}^{;0} - 2\,\Box_3 R_3 - \frac{1}{2} R_3^2 + 2R_3 R_{30}^0\right)$$
$$+ \frac{1}{180\tilde{m}_0}\left(-\frac{1}{2} R_3^{\alpha'\beta'\gamma'\delta'} R_{3\alpha'\beta'\gamma'\delta'} + 2R_{30\alpha'\beta'\gamma'} R_{30}^{\alpha'\beta'\gamma'} - 3\,\Box_3 R_{30}^0\right.$$
$$+ 2R_{3;0}^{;0} - 4R_{3\alpha'}^0 R_{30}^{\alpha'} - 4R_3^{\alpha'\beta'} R_{3\alpha'0\beta'}^0 - 2R_{3;0\alpha'}^{\alpha'0}$$
$$\left. + \frac{1}{2}\Box_3 R_3 - 2R_3^{\alpha'0} R_{3\alpha'0} + \frac{1}{2} R_3^{\alpha'\beta'} R_{3\alpha'\beta'}\right)$$
$$= \frac{1}{2}\left[\frac{11}{G_3 b}\left(\frac{db}{dt}\right)^2 - \frac{4\tilde{m}_0^3}{3} + \frac{1}{2\tilde{m}_0}\left(\xi - \frac{1}{2}\right)^2\left(- \frac{1}{2} R_3^2 + 2R_3 R_{30}^0\right)\right.$$
$$+ \frac{1}{180\tilde{m}_0}\left(- \frac{1}{2} R_3^{\alpha'\beta'\gamma'\delta'} R_{3\alpha'\beta'\gamma'\delta'} + 2R_{3\alpha'\beta'\gamma'}^0 R_{30}^{\alpha'\beta'\gamma'}\right.$$
$$- 4R_{3\alpha'}^0 R_0^{\alpha'} - 4R_3^{\alpha'\beta'} R_{3\alpha'0\beta'}^0 - 2R_{3;0\alpha'}^{\alpha'0}$$

$$+ 4\,\Box_3 R_3 - 2R_3^{\alpha'0}R_{3\alpha'0} + \frac{1}{2}R_3^{\alpha'\beta'}R_{3\alpha'\beta'}\Big)$$

$$= \frac{1}{2}\left[\frac{11}{G_3 b}\left(\frac{db}{dt}\right)^2 - \frac{4}{3}\tilde{m}_0^3\right] + \frac{1}{\tilde{m}_0}\left[4\left(\xi - \frac{1}{6}\right)^2 - \frac{53}{180}\right]a^{-2}\frac{d^2a}{dt^2}$$

$$+ \frac{1}{45\tilde{m}_0}\left[3a^{-1}\frac{d^4a}{dt^4} + 2a^{-2}\frac{da}{dt}\frac{d^3a}{dt^3} - 2a^{-3}\left(\frac{da}{dt}\right)^2\frac{d^2a}{dt^2}\right] \tag{6.4a}$$

$$p_1 = p_2 = \left[\frac{b}{2G_3} - 2\tilde{m}_0\left(\frac{1}{6} - \xi\right)\right]\left(R^1_{(3)1} - \frac{1}{2}R_3\right)$$

$$+ \frac{1}{2}\left[\frac{11}{G_3 b}\left(\frac{db}{dt}\right)^2 - \frac{4}{3}\tilde{m}_0^3\right] + \frac{1}{2\tilde{m}_0}\left(\xi - \frac{1}{6}\right)^2\Big(2R_{3;1}^{;1}$$

$$- 2\,\Box_3 R_3 - \frac{1}{2}R_3^2 + 2R_3 R^1_{(3)1}\Big)$$

$$+ \frac{1}{180\tilde{m}_0}\Big[-\frac{1}{2}R_3^{\alpha'\beta'\gamma'\delta'}R_{3\alpha'\beta'\gamma'\delta'} + 2R_{(3)1\alpha'\beta'\gamma'}R_3^{1\alpha'\beta'\gamma'}$$

$$- 4\,\Box_3 R^1_{(3)1} + 2R_{3;1}^{;1} - 4R^1_{(3)\alpha}R^\alpha_{(3)1} - 4R_3^{\alpha 1 3}R^1_{3\alpha 1\beta}$$

$$- 2R^{\alpha 1}_{3;\alpha 1} + \Box_3 R^1_{(3)1} + \frac{1}{2}\Box_3 R_3 - 2R_3^{\alpha 1}R_{3\alpha 1} + \frac{1}{2}R_3^{\alpha\beta}R_{3\alpha\beta}\Big]$$

$$= \left[2\tilde{m}_0\left(\frac{1}{6} - \xi\right) - \frac{b}{2G_3}\right]a^{-1}\frac{d^2a}{dt^2} + \frac{1}{2}\left[\frac{11}{G_3 b}\left(\frac{db}{dt}\right)^2 - \frac{4}{3}\tilde{m}_0^3\right]$$

$$+ \frac{4}{\tilde{m}_0}\left(\xi - \frac{1}{6}\right)^2\left(a^{-1}\frac{d^4a}{dt^4} - a^{-2}\frac{d^2a}{dt^2}\right)$$

$$+ \frac{1}{180\tilde{m}_0}\left[7a^{-1}\frac{d^4a}{dt^4} - 22a^{-2}\left(\frac{d^2a}{dt^2}\right)^2\right] \tag{6.4b}$$

where b is given by equation (2.2) and $\tilde{m}_0$ is defined through (3.12c).

Taking the trace of equation (6.3), one obtains

$$-\frac{1}{2}\left[\frac{b}{2G_3} - 2\tilde{m}_0\left(\frac{1}{6} - \xi\right)\right]R_3 + \frac{3}{2}\left[\frac{11}{G_3 b}\left(\frac{db}{dt}\right)^2 - \frac{4}{3}\tilde{m}_0^3\right]$$
$$+ \frac{1}{2\tilde{m}_0}\left(\xi - \frac{1}{6}\right)^2\left(-4\,\Box_3 R_3 + \frac{1}{2}R_3^2\right) + \frac{1}{180\tilde{m}_0}\left(\frac{1}{2}R_3^{\alpha'\beta'\gamma'\delta'}R_{3\alpha'\beta'\gamma'\delta'}\right.$$
$$\left. - \frac{1}{2}\Box_3 R_3 - \frac{1}{2}R_3^{\mu'\nu'}R_{3\mu'\nu'}\right) = -8\pi\langle T\rangle \tag{6.5}$$

In the case $\langle T\rangle = 0$, equation (6.5) yields

$$-\frac{1}{2}\left[\frac{b}{2G_3} - 2\tilde{m}_0\left(\frac{1}{6} - \xi\right)\right]R_3 + \frac{3}{2}\left[\frac{11}{G_3 b}\left(\frac{db}{dt}\right)^2 - \frac{4}{3}\tilde{m}_0^3\right]$$
$$+ \frac{1}{2\tilde{m}_0}\left(\xi - \frac{1}{6}\right)^2\left(-4\,\Box_3 R_3 + \frac{1}{2}R_3^2\right) + \frac{1}{180\tilde{m}_0}\left(\frac{1}{2}R_3^{\alpha'\beta'\gamma'\delta'}R_{3\alpha'\beta'\gamma'\delta'}\right.$$
$$\left. - \frac{1}{2}\Box_3 R_3 - \frac{1}{2}R_3^{\mu'\nu'}R_{3\mu'\nu'}\right) = 0 \tag{6.6}$$

Using the definitions of $b(t)$ and $\tilde{m}_0$ given by equations (2.23) and (3.12c), respectively, and taking the limit $t \to \infty$, one obtains

$$m_0 = 0 \tag{6.7}$$

Thus if the energy-momentum tensor is traceless, the effective cosmological constant given by equation (5.8) vanishes at late times.

REFERENCES

Bateman, H. (1955). *Higher Transcendental Functions,* Vol. III, McGraw-Hill, New York.

Dadhich, N., and Patel, L. K. (1993). IUCAA Preprint 19/93.

Fujiwara, Y., Higuchi, S., Hosoya, A., Mishima, T., and Siino, M. (1991). *Physical Review D,* **44**, 1763.

Hawking, S. W., and Ellis, G. F. R. (1973). *The Large Scale Structure of Space-Time,* Cambridge University Press, Cambridge.

Mann, R. B., Tarasov, L., Mckeon, D. G. C., and Steele, T. (1988/89). *Nuclear Physics B,* **311**, 630.

McGuigan, M. (1991). *Physical Review D,* **43**, 1199.

Ruiz, E., and Senovilla, J. M. M. (1922). *Physical Review D,* **45**, 1995.

Souradeep, T., and Sahani, V. (1992). *Physical Review D,* **46**, 1616.

Senovilla, J. M. M. (1990). *Physical Review Letters,* **64**, 2219.

Srivastava, S. K. (1992a). *International Journal of Theoretical Physics,* **31**,

Srivastava, S. K. (1992b). *Journal of Mathematical Physics,* **33**, 3117.

Srivastava, S. K. (1993). *Classical Quantum Gravity,* **10**, 2307.